GROUP A

1. Find all real solutions (x_1, x_2, x_3, λ) for the system of equations

$$x_2 - 3x_3 - x_1\lambda = 0,$$
$$x_1 - 3x_3 - x_2\lambda = 0,$$
$$x_1 + x_2 + x_3\lambda = 0.$$

2. Let $\{x_n\}_{n\geq 1}$ be a sequence defined by $x_1 = 1$ and

$$x_{n+1} = \left(x_n^3 + \frac{1}{n(n+1)(n+2)}\right)^{1/3}, \quad n \geq 1.$$

Show that $\{x_n\}_{n\geq 1}$ converges and find its limit.

3. Consider all permutations of the integers $1, 2, \ldots, 100$. In how many of these permutations will the 25th number be the minimum of the first 25 numbers and the 50th number be the minimum of the first 50 numbers?

GROUP B

4. An urn contains $r > 0$ red balls and $b > 0$ black balls. A ball is drawn at random from the urn, its colour noted, and returned to the urn. Further, $c > 0$ additional balls of the same colour are added to the urn. This process of drawing a ball and adding c balls of the same colour is continued. Define $X_i = 1$ if at the i-th draw the colour of the ball drawn is red, and 0 otherwise. Compute $\mathrm{E}(\sum_{i=1}^{n} X_i)$.

5. Suppose X_1 and X_2 are identically distributed random variables, not necessarily independent, taking values in $\{1, 2\}$. If $\mathrm{E}(X_1X_2) = 7/3$ and $\mathrm{E}(X_1) = 3/2$, obtain the joint distribution of (X_1, X_2).

6. A fair 6-sided die is rolled repeatedly until a 6 is obtained. Find the expected number of rolls conditioned on the event that none of the rolls yielded an odd number.

7. Suppose $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ is a random sample from a bivariate normal distribution with $\mathrm{E}(X_i) = \mathrm{E}(Y_i) = 0$, $\mathrm{Var}(X_i) = \mathrm{Var}(Y_i) = 1$ and unknown $\mathrm{Corr}(X_i, Y_i) = \rho \in (-1, 1)$, for all $i = 1, \ldots, n$. Define $W_n = \frac{1}{n}\sum_{i=1}^{n} X_i Y_i$.

   (a) Is W_n an unbiased estimator of ρ? Justify your answer.

   (b) For large n, obtain an approximate level $(1-\alpha)$ two-sided confidence interval for ρ, where $0 < \alpha < 1$.

8. Let $\{X_1, \ldots, X_n\}$ be an i.i.d. sample from $f(x : \theta)$, $\theta \in \{0, 1\}$, with

$$f(x:0) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x:1) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Based on the above sample, obtain the most powerful test for testing $H_0 : \theta = 0$ against $H_1 : \theta = 1$, at level α, with $0 < \alpha < 1$. Find the critical region in terms of the quantiles of a standard distribution.

9. Suppose (y_i, x_i) satisfies the regression model,

$$y_i = \alpha + \beta x_i + \epsilon_i, \quad \text{for } i = 1, \ldots, n,$$

   where $\{x_i : 1 \leq i \leq n\}$ are fixed constants and $\{\epsilon_i : 1 \leq i \leq n\}$ are i.i.d. $N(0, \sigma^2)$ errors, where α, β and $\sigma^2 (> 0)$ are unknown parameters.

   (a) Let $\widetilde{\alpha}$ denote the least squares estimate of α obtained assuming $\beta = 5$. Find the mean squared error (MSE) of $\widetilde{\alpha}$ in terms of the model parameters.

   (b) Obtain the maximum likelihood estimator of this MSE.